Modular Gait Optimization: From Unit Moves to Multi-Step Trajectory in Bipedal Systems

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***Abstract*—This report addresses the computational challenges in hybrid direct collocation (HDC) for multi-step bipedal systems trajectory optimization. We introduce the Gait Modularization and Optimization Technique (GMOT), which utilizes unit gait trajectories as initialization for multi-step trajectory optimization, leading to significant improvements in efficacy and efficiency over naive HDC methods. We evaluated our results against three gaits: bunny hopping, walking, and running. The animated images and codes are available at https://github.com/QingquanBao/2DBiped.**

***Index Terms*—Hybrid System, Constrained Direct Collocation, Bipedal System, Biped Gait**

I. Introduction

Trajectory optimization in bipedal robotics, while progressively maturing, still confronts significant challenges, particularly when utilizing hybrid direct collocation (HDC) methods. While HDC has demonstrated considerable success in enabling dynamic and complex locomotion, issues such as initialization sensitivity, constraint management, and precise gait definition critically influence the efficiency and outcome of the optimization process.

In response to these challenges, this report introduces a structured and modular approach to Multi-step trajectory optimization using collocation, which we call the gait Modularization and Optimization Technique (GMOT). GMOT not only streamlines the optimization of bipedal gaits but also significantly improves computational efficiency. We apply this methodology to three fundamental gaits—bunny hopping, walking, and running—and provide a comparative analysis against traditional HDC approaches, showcasing our method's superior performance.

Key contributions of this report include the development of GMOT building upon the DIRCON algorithm [1] and the identification of empirical best practices. We achieve more efficient optimization with more realistic biped gaits, offering practical insights and guidelines for future applications.

II. Background

A. Biped Model Setup

Our study focuses on a basic five-joint bipedal model with actuation in four of the joints, as depicted in Fig. 1. The joint configuration space is denoted as $\mathbf{q} = [x, z, \theta, q_1, q_2, q_3, q_4]^T \in \mathbb{R}^7$, representing the biped's position and orientation in the plane, along with its internal joint angles. The actuator input space is defined as $\mathbf{u} = [u_1, u_2, u_3, u_4]^T \in \mathbb{R}^4$, corresponding to the torques applied at the four actuated joints. The complete state space, combining configuration and velocity, is represented as $\mathbf{x} = [\mathbf{q}, \mathbf{v}]^T \in \mathbb{R}^{14}$ with $\mathbf{v} = \dot{\mathbf{q}}$.

B. Hybrid Trajectory Optimization

In our study, we denote the trajectory of the bipedal system through a sequence of modes and knot points as $x_1^1, ..., x_{N^1}^1, ..., x_1^j, ..., x_{N^j}^j, ..., x_{N^M}^M$, where M is the mode sequence length, j indexes the mode sequence, and N^j refers to the Nth knot point within the jth mode. This notation allows us to describe the state of the system at any given instance within the hybrid trajectory optimization framework.

Hybrid trajectory optimization integrates the continuous dynamics of walking phases with discrete transitions, facilitated by guards and resets. These transitions enable the biped model to switch seamlessly between locomotive modes. The guard function, such as the height of a foot $\phi(q)_y = 0$, signals the need for a mode transition, while the reset map updates the state $\mathbf{q}$ and $\mathbf{v}$, according to the impact dynamics:

$$\begin{aligned} \mathbf{q}_1^j &= \mathbf{q}_{N^{j-1}}^{j-1} \\ \mathbf{v}_1^j &= \mathbf{v}_{N^{j-1}}^{j-1} + \mathbf{M}^{-1}\mathbf{J}^T\mathbf{\Lambda}^{j-1}, \end{aligned} \tag{1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{J}$ is the Jacobian of the contact constraints, and $\mathbf{\Lambda}$ represents the impulse due to contact.

C. Contact Constrained Dynamics

The dynamics of the biped are governed by the contact constraints, which ensure that the feet maintain contact with the ground. The constraints are mathematically formulated as:

$$\begin{aligned} \phi(q(t)) &= 0 \\ \psi(q,v) \equiv \frac{\mathrm{d}\phi}{\mathrm{d}t} &= J(q)v = 0 \\ \alpha(q,v,u,\lambda) \equiv \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} &= \frac{\mathrm{d}J(q)}{\mathrm{d}t}v + J(q)\bar{f}(x,u,\lambda) = 0 \end{aligned} \tag{2}$$

To address the complexity of an over-constrained system, constrained direct collocation is employed to manage the manifold constraints from the biped's interaction with the environment.

D. Constrained Direct Collocation

The optimization problem is structured using constrained direct collocation (DIRCON) [1], aiming to minimize the cost function over the trajectory while satisfying the system's dynamic and contact constraints:

$$\begin{aligned} \min_z \quad & \ell_f(X_N) + h\sum_{k=1}^{N}\ell(x_k,u_k) \\ \text{s.t.} \quad & 0 = \bar{g}(x_k,u_k,\lambda_k,x_{k+1},u_{k+1},\lambda_{k+1},\bar{\lambda}_k,\bar{\gamma}_k) \\ & \quad \text{for} \quad k = 1,...,N-1 \\ & 0 = \phi(q_k) = \psi(x_k) = \alpha(q_k,v_k,u_k,\lambda_k) \\ & \quad \text{for} \quad k = 1,...,N \\ & 0 \geq m(z), \end{aligned} \tag{3}$$

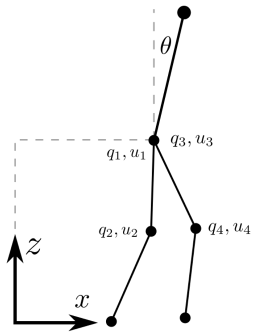


Fig. 1: Simple Biped Model

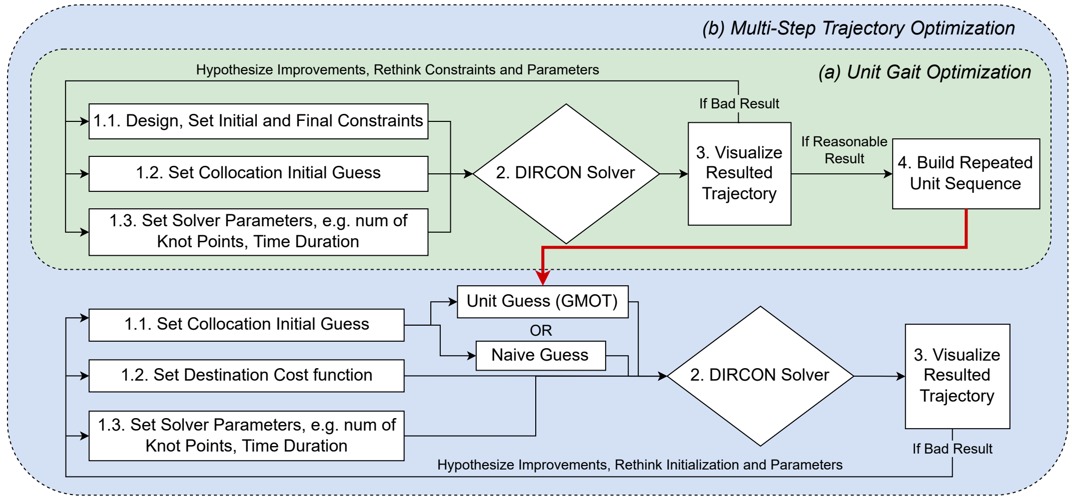


Fig. 2: **GMOT Overview**. (a) The workflow in optimizing for a unit gait trajectory. (b) The method in optimizing for a multi-step trajectory, where the red bold arrow indicates our GMOT method.

where the collocation constraint ensures the continuity and consistency of the motion between collocation points:

$$\bar{g}(x_k, u_k, \lambda_k, x_{k+1}, u_{k+1}, \lambda_{k+1}, \bar{\lambda}_k, \bar{\gamma}_k) = \dot{x}_s(t_k + .5h) - \begin{bmatrix} v_c + J(q_c)^T \bar{\gamma}_k \\ \bar{f}(x_c, u_c, \bar{\lambda}_k \end{bmatrix}. \quad (4)$$

III. Methods

In this section, we detail our Gait Modularization and Optimization Technique (GMOT). First, we define three biped gaits, i.e., walking, running, and bunny hopping. Then, we introduce the approaches to optimize a unit gait, and finally solve the multi-step trajectory with the initialization of repeated sequential units.

A. Definitions: Unit & Full Trajectory, Mode & Gaits

Unit Trajectory: In the context of a biped system, we assume that most ideal biped gaits follow some form of periodic sequence when excluding translational motion. We define a *unit trajectory* to be *one period in that motion sequence*. In terms of states, we simply define a unit trajectory to be $x_1^1, ..., x_{N^1}^1$.

Multi-step Trajectory: Contrast to the unit trajectory, which aims to be a minimum repeatable sequence of motion that defines a gait, we define a *multi-step trajectory of gait* α to be a *free-form sequence of gait* α *that moves towards a distance target*. In terms of states, we simply define a multi-step trajectory to be $x_1^1, ..., x_{N^1}^1, ..., x_1^j, ..., x_{N^j}^j, ..., x_{N^M}^M$.

Mode & Gaits: In the context of hybrid systems, we define distinct unit modes in a biped system to facilitate further modeling. These modes encapsulate all potential contact dynamics a biped may encounter:

- Right Stance: right foot contacts the ground while left foot in the air;
- Left Stance: left foot contacts the ground while right foot in the air;
- Double Stance: both feet contact the ground;
- Flight Phase: both feet in the air, i.e., no contact constraints in this mode.

The well-defined modes enable us to construct gait units that are intuitive, concise, and optimal for rapid optimization. The gait unit should satisfy several properties: (1) align with the human intuitive understanding of gaits; (2) Short and simple enough for fast optimization. The unit is defined as a predefined mode sequence based on human prior:

- Walk: [Right Stance, Left Stance],
- Run: [Right Stance, Flight Phase, Left Stance, Flight Phase],
- Bunny hop: [Double Stance, Flight Phase, Double Stance].

B. Unit Gaits Optimization

To create a repeatable gait unit, we use the workflow depicted in Fig. 2. We impose symmetrical initial and final constraints on $\mathbf{x}$, which we will refer to as $\mathbf{x}_0$ and $\mathbf{x}_f$ respectively, and use DIRCON to solve for the motion in between.

For any gait, we would have to design $\mathbf{x}_0$ and $\mathbf{x}_f$'s $\mathbf{q}$ component using forward and inverse kinematics, and provide educated guesses for $\mathbf{v}$. This process can be thought of as hypothesizing a feasible initial and final constraint and is often constantly iterated for improvements. We will discuss two representative gaits for unit gait optimization, bunny hopping and walking.

For a single hopping unit trajectory $\mathbf{x}_0^{hop}, ..., \mathbf{x}_f^{hop}$, it has a simple symmetry constraint due to the same mode repetition when repeating the unit, while translating some distance $\mathbf{d}$:

$$\mathbf{x}_0^{hop}[1:7] = \mathbf{x}_f^{hop}[1:7] \quad \mathbf{x}_0^{hop}[0] + \mathbf{d} = \mathbf{x}_f^{hop}[0]. \quad (5)$$

For a single walking unit $\mathbf{x}_0^{(r)}, ..., \mathbf{x}_f^{(r)}, \mathbf{x}_0^{(l)}, ..., \mathbf{x}_f^{(l)}$, since the walking gait is left-right leg symmetric, we can constrain and solve for the unit trajectory on one leg. Here, we choose to constrain and solve for the right stance unit, translating some distance $\mathbf{d}$ forward, which gives the following constraint,

$$
\begin{aligned}
&\mathbf{q}_0^{(r)}[0] + \mathbf{d} = \mathbf{q}_f^{(r)}[0] \\
\mathbf{q}_0^{(r)}[1:3] = \mathbf{q}_f^{(r)}[1:3] \quad &\mathbf{v}_0^{(r)}[1:3] = \mathbf{v}_f^{(r)}[1:3] \\
\mathbf{q}_0^{(r)}[3:5] = \mathbf{q}_f^{(r)}[5:7] \quad &\mathbf{v}_0^{(r)}[3:5] = \mathbf{v}_f^{(r)}[5:7]
\end{aligned} \tag{6}
$$

After getting the nominal unit trajectory $\hat{\mathbf{x}}_0^{(r)}, ..., \hat{\mathbf{x}}_f^{(r)}$, we can construct the other nominal stance unit trajectory $\hat{\mathbf{x}}_0^{(l)}, ..., \hat{\mathbf{x}}_f^{(l)}$ with a symmetric transformation, i.e., for any $i \in [0, N]$,

$$
\begin{aligned}
&\hat{\mathbf{q}}_i^{(l)}[0] = \hat{\mathbf{q}}_i^{(r)}[0] + \mathbf{d} \\
\hat{\mathbf{q}}_i^{(l)}[1:3] = \hat{\mathbf{q}}_i^{(r)}[1:3] \quad &\hat{\mathbf{v}}_i^{(l)}[1:3] = \hat{\mathbf{v}}_i^{(r)}[1:3] \\
\hat{\mathbf{q}}_i^{(l)}[3:5] = \hat{\mathbf{q}}_i^{(r)}[5:7] \quad &\hat{\mathbf{v}}_i^{(l)}[3:5] = \hat{\mathbf{v}}_i^{(r)}[5:7] \\
\hat{\mathbf{q}}_i^{(l)}[5:7] = \hat{\mathbf{q}}_i^{(r)}[3:5] \quad &\hat{\mathbf{v}}_i^{(l)}[5:7] = \hat{\mathbf{v}}_i^{(r)}[3:5]
\end{aligned} \tag{7}
$$

C. Multi-step Trajectory Optimization with Gait Module as Initialization

Using repeated unit trajectories as a final planned trajectory has drawbacks: (1) Designing feasible velocity constraints is extremely challenging compared to positional constraints if considering high-speed movements; (2) Inability of the solver to find suitable solutions for unit trajectories leads to discontinuities and dynamic infidelities, resulting in error accumulation over longer trajectories. However, optimizing a unit gait, given its physical simplicity, is relatively manageable.

Conversely, employing multi-step trajectory optimization with naïve initialization and cost functions (like effort penalties, and destination costs) makes solving the optimization problem exceedingly intractable. Yet, the advantage of this approach lies in its ability to produce smoother, more flexible solutions over longer horizons.

As illustrated in Fig. 2, GMOT merges the strengths of two methods by using a sequence of unit gaits as the initialization in multi-step trajectory optimization. This technique enables the solver to generate smoother, more efficient multi-step gait results, combining simplicity with long-term optimization efficiency.

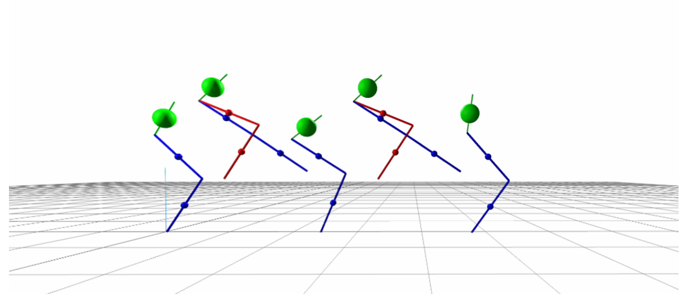
IV. Experiments & Empirical Discussion

A. Qualitative and Intuitive Evaluation Criteria

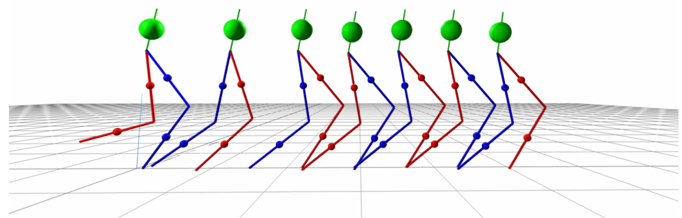
Our evaluation of bipedal gaits includes qualitative criteria to ensure they not only meet technical metrics but also align with human movement patterns:

- Human-like Movement: Gaits should closely resemble how humans or other bipeds move.
- Minimal Extraneous Movements: Optimized gaits should avoid unnecessary actuations.
- Smoothness: The movement should be fluid and continuous, avoiding unnatural abrupt changes.

These criteria are assessed through reviews of simulations and discussed below.



(a) Repeated bunny hopping units



(b) Repeated walking units

Fig. 3: Trajectory of different gaits by repeating optimized gait units

B. Unit Trajectory Optimization

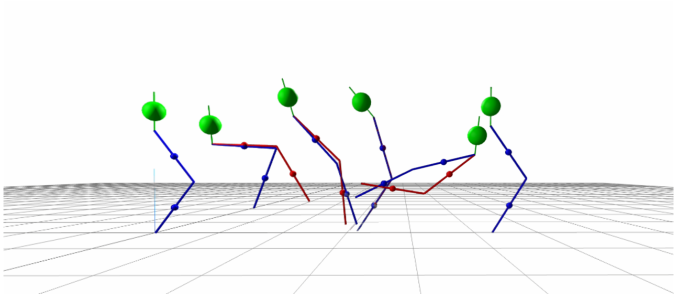
As illustrated in Fig. 3, we have visualized repeated unit trajectories for (a) hopping and (b) walking gaits. Through precise state constraint design, these unit gaits effectively mimicked human movements. However, this approach was not without issues. Notably, extraneous motions, such as leg shaking in the walking gait, led to less smooth trajectories. Furthermore, the process was hindered by frequent solver errors due to infeasible constraints and required significant effort in designing appropriate initial and final state constraints.

C. Multi-step Trajectory Optimization

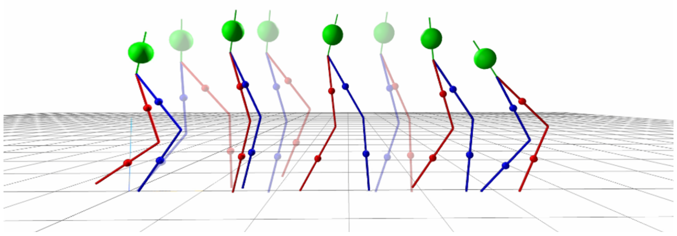
Figure 4 displays the trajectories for various gaits optimized through our GMOT methodology. We initialized hopping and walking with their respective repeated unit trajectories, while for running, we utilized repeated walking units as a starting point.

The GMOT approach effectively adjusts for variable velocities and arrival distances, yielding trajectories that are more aligned with human motion. These trajectories are characterized by minimal extraneous movements and exhibit exceptional smoothness. The walking and running gaits optimized by GMOT show great promise, although the jumping trajectory appears somewhat counter-intuitive in parts.

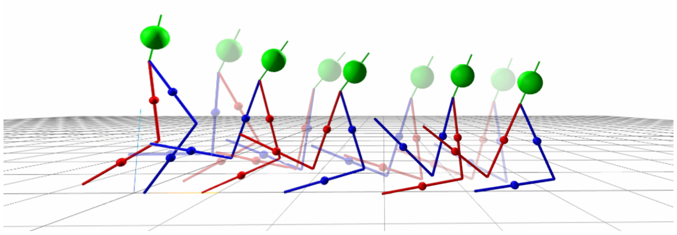
A notable observation is the inconsistency in step lengths. For example, the biped might reach the intended destination using fewer steps than planned, resulting in unnecessary actions in the remaining steps. This highlights that while GMOT significantly improves the optimization of bipedal gaits, maintaining control over intermediate states without additional constraints is a challenge. Implementing such constraints is possible but requires significant effort and meticulous cali-



(a) Bunny hopping generated by GMOT



(b) Walking generated by GMOT



(c) Running generated by GMOT

Fig. 4: Trajecotry of different gaits generated by GMOT

bration. Future research should consider exploring automated methods for constraint generation to streamline this process.

D. Comparison of Time Efficiency

In assessing the time efficiency of various trajectory optimization methods, we define an efficiency metric **solving time per collocation knot point over** 10^4 **iterations**. This metric serves as a proxy of time cost, with optimization complexity factored out. We compared our GMOT method against a baseline approach, which utilizes multi-step trajectory optimization with naive initial guesses (including a static stance and uniform positional transition towards the target). The results, as shown in Table I, demonstrate that GMOT significantly outperforms the baseline in solving time across all gaits.

It is important to note a few observations that are not directly evident from the table. Firstly, for the baseline and unit optimization methods, the number of iterations is capped below 7×10^4 due to the computational burden becoming prohibitive beyond this point—often resulting in an inability to derive solutions within an hour (in Apple Silicon M1 pro). In contrast, the GMOT method is capable of solving well-

Gait	Method	Traj length	iters (1e4)	Time (s)	Efficiency
Hopping	Baseline	36	4	827	5.743
	Unit	18	5	132	1.467
	GMOT	36	20	335	0.465
Walking	Baseline	64	15	385	0.401
	Unit	8	0.4	55	17.187
	GMOT	64	20	448	0.350
Running	Baseline	64	6	296	0.771
	GMOT	64	9	361	0.626

TABLE I: Time efficiency comparison among different optimization methods. Efficiency is measured as the time cost per 10^4 iterations for each trajectory knot point

tuned trajectories even with prolonged iterations. Additionally, the baseline method does not consistently yield aesthetically pleasing or functionally optimal trajectories, highlighting a limitation in its applicability for more complex gait optimizations.

E. Discussion on slack variables γ

In constrained direct collocation, slack variables γ play a pivotal role in managing constraints and ensuring feasible solutions. However, our experiments revealed challenges when strictly adhering to the approach outlined in [1], particularly in terms of trajectory optimization stability.

To address this, we introduced a novel constraint on the slack variables. Drawing inspiration from physical principles, we implemented a cone constraint similar to frictional forces, expressed as $|\gamma_x| \leq \mu\gamma_z$. This decision was driven by the intuition that the velocity correction, represented by γ, should be on a similar scale to the frictional forces experienced by the biped.

V. Future Work

Our future endeavors include the formal design of experiments to test our methodology against a range of quantitative metrics in diverse scenarios. We aim to explore the application of our initialization and constraining techniques to other direct optimization methods, like multiple shooting, to assess their broader applicability. Additionally, we plan to challenge our model with more complex gaits, such as backflipping, to further test its capabilities. A particularly promising direction is automating constraint design to more closely mirror human locomotion intuitively, bridging the gap between robotic movement and natural human gait patterns.

Acknowledgment

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References

[1] Michael Posa, Scott Kuindersma, and Russ Tedrake. Optimization and stabilization of trajectories for constrained dynamical systems. *2016 IEEE International Conference on Robotics and Automation (ICRA)*, pages 1366–1373, 2016. 1, 4